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BOUNDED SOLUTIONS OF DIFFERENCE EQUATIONS IN A BANACH SPACE WITH ASYMPTOTICALLY CONSTANT OPERATOR COEFFICIENT

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We study the problem of existence of a unique bounded solution for a difference equation on the half axis with asymptotically constant operator coefficients in a Banach space. Necessary and sufficient conditions for the existence and uniqueness of bounded solutions are obtained for equations with and without initial condition.

Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space, let $L(X)$ be a subspace of linear continuous operators in X, and let $I \in L(X)$ be the identity operator. By $\sigma(A)$ we denote the spectrum of the operator $A \in L(X)$. We use the term "subspace" for a closed linear subset of X.

Consider a difference equation on the semiaxis

$$x_{n+1} = A_n x_n + y_n, \quad n \geq 0, \tag{1}$$

where $\{A_n : n \geq 0\} \subset L(X)$ and $\{y_n \mid n \geq 0\} \subset X$ are known sequences and $\{x_n \mid n \geq 0\} \subset X$ is the required sequence.

Equations of this type were investigated by numerous leading experts in the field of differential equations, including, in particular, Samoilenko, Boichuk, Dorogovtsev, Slyusarchuk, Horodnii, Pokutnyi, and many others (see, e.g., [1–5] and the references therein).

Equations with constant coefficients are studied most comprehensively. In particular, the following results are well known for the bounded solutions of Eq. (1) in the case of a constant operator coefficient:

Theorem 1 [5, 6]. *The difference equation (1) with $A_n = A$, $n \geq 0$, has a unique bounded solution $\{x_n \mid n \geq 0\} \subset X$ for any bounded sequence $\{y_n \mid n \geq 0\} \subset X$ if and only if*

$$\sigma(A) \subset \{z \in \mathbf{C} \mid |z| > 1\}.$$

Theorem 2 [7]. *The difference equation (1) with $A_n = A$, $n \geq 0$, has a bounded solution $\{x_n \mid n \geq 1\} \subset X$ for any bounded sequence $\{y_n \mid n \geq 0\} \subset X$ and any $x_0 \in X$ if and only if*

$$\sigma(A) \subset \{z \in \mathbf{C} \mid |z| < 1\}.$$

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In the general case of variable operator coefficient, the conditions of existence and uniqueness of a bounded solution are connected with the condition of discrete exponential dichotomy (see, e.g., [8]). The problem of verification of these conditions is quite complicated. Therefore, the investigation of special cases is very important. As one of these cases, we can mention the case of piecewise constant operator coefficients for equations on the entire axis whose investigation was originated in [9, 10].

In the present paper, we consider another important case in which the coefficient asymptotically approaches a constant operator. Namely, we study the problem of existence and uniqueness of a bounded solution of Eq. (1) in the case of a variable operator, which is asymptotically stable at infinity.

The following definition describes the exact conditions for the asymptotic behavior of operators at infinity:

Definition 1. *By $As(X)$ we denote the class of all sequences $\{B_n : n \geq 1\} \subset L(X)$ satisfying the following conditions:*

(i) *there exists an operator $B \in L(X)$ such that (in the matrix form)*

$$B_n \to B, \quad n \to +\infty;$$

(ii) $\forall n \geq 0 \colon \exists B_n^{-1} \in L(X)$;

(iii) $\forall n \geq 0 \colon B_n B = B B_n$;

(iv) $\exists B^{-1} \in L(X)$;

(v) $\exists C > 0 \colon \forall m, n \geq 0, m < n \colon \prod_{k=m}^{n} \left\| B^{-1} B_k \right\| \leq C$;

(vi) $\exists C > 0 \colon \forall m, n \geq 0, m < n \colon \prod_{k=m}^{n} \left\| B_k^{-1} B \right\| \leq C$.

Equation without Initial Conditions

By

$$l_\infty(X) := \left\{ (x_1, x_2, \ldots) \;\middle|\; \sup_{n \geq 1} |x_n| < +\infty \right\}$$

we denote a Banach space with uniform norm.

Lemma 1. *Suppose that $C \in L(l_\infty(X))$ and a sequence $\{A_n : n \geq 1\} \subset L(X)$ has (in the operator form) the limit $A \in L(X)$. Then, for any bounded sequence $v \in l_\infty(X)$, there exist $N \in \mathbf{N}$ and a bounded sequence $u \in l_\infty(X)$ such that*

$$(A_n - A)(Cu)_n + u_n = v_n, \quad n \geq N. \tag{2}$$

Proof. Let C be a nonzero operator (otherwise, the statement is obvious). In view of the asymptotic behavior of the operator coefficient in condition (i), we can take a sufficiently large $N \in \mathbf{N}$ such that

$$\forall n \geq N: \quad \|A_n - A\| \|C\| \leq \frac{1}{2}.$$

We seek the required sequence u in a subspace

$$l_\infty(X,N) := \left\{ \left(\underbrace{\bar{0},\ldots,\bar{0}}_{N-1}, x_N, x_{N+1}, \ldots \right) \,\middle|\, x \in l_\infty(X) \right\}$$

of the space $l_\infty(X)$.

Denote

$$(Dx)_n = (A_n - A)x_n, \qquad n \geq N.$$

Thus, D is a linear continuous operator in the space $l_\infty(X,N)$ whose norm does not exceed $\dfrac{1}{2\|C\|}$.

Let P_N be a projector from the space $l_\infty(X)$ onto the subspace $l_\infty(X,N)$, which transforms the first $N-1$ coordinates into zero elements. Denote $C_N = P_N C$. Then C_N is a linear bounded operator in the space $l_\infty(X,N)$ whose norm does not exceed $\|C\|$.

We can now rewrite Eq. (2) as an equation in the space $l_\infty(X,N)$ of the form

$$DC_N u + u = P_N v.$$

Since

$$\|DC_N\| \leq \frac{1}{2\|C\|}\,\|C\| = \frac{1}{2} < 1,$$

this equation has a unique solution $u \in l_\infty(X,N)$.

Lemma 2. *Suppose that the sequence $\{A_n : n \geq 1\}$ belongs to the class $As(X)$ and has the limit $A \in L(X)$. Denote*

$$R(n,m) := A_n A_{n-1} \ldots A_{m+1}, \quad n > m \geq -1, \quad R(n,n) = I, \quad n \geq -1,$$

$$R(m,n) := (R(n,m))^{-1}, \quad n > m \geq -1.$$

An arbitrary solution of Eq. (1) has the form

$$x_n = R(n-1,-1)x_0 + R(n-1,0)y_0 + R(n-1,1)y_1 + \ldots + R(n-1,n-1)y_{n-1}, \quad n \geq 1,$$

and there exists a constant $L > 0$ such that

$$\forall m \geq -1,\ n \geq -1: \quad L^{-1} \leq \|R(n,m)A^{m-n}\| \leq L.$$

Proof. By induction, we can easily show that the proposed formula for the solution is true. In view of conditions (v) and (vi) from the definition of the class $As(X)$, there exists a constant $C > 1$ such that

$$\forall n \geq 0: \quad \prod_{m=0}^{n-1} \left\| A_m^{-1} A \right\| \in [1/C, C],$$

$$\forall n \geq 0: \quad \prod_{m=0}^{n-1} \left\| A^{-1} A_m \right\| \in [1/C, C].$$

Thus, for $n > m \geq -1$, we get

$$\|R(n,m)A^{m-n}\| = \left\|A_n A^{-1} \dots A_{m+1} A^{-1}\right\| \leq \prod_{k=m+1}^{n} \left\|A_k A^{-1}\right\| \leq C^2,$$

$$\|R(m,n)A^{n-m}\| = \left\|A_{m+1}^{-1} A \dots A_n^{-1} A\right\| \leq \prod_{k=m+1}^{n} \left\|A_k^{-1} A\right\| \leq C^2,$$

$$\|R(n,m)A^{m-n}\| \geq \|R(m,n)A^{n-m}\|^{-1} \geq 1/C^2,$$

$$\|R(m,n)A^{n-m}\| \geq \|R(n,m)A^{m-n}\|^{-1} \geq 1/C^2,$$

and we can set $L = C^2$.

Theorem 3. *Suppose that the sequence $\{A_n : n \geq 1\}$ belongs to the class $As(X)$ and has the limit $A \in L(X)$. The difference equation (1) has a unique bounded solution $\{x_n \mid n \geq 0\} \subset X$ for any bounded sequence $\{y_n \mid n \geq 0\} \subset X$ if and only if*

$$\sigma(A) \subset \{z \in \mathbf{C} \mid |z| > 1\}.$$

Proof. *Necessity*. Assume that, for any bounded sequence $\{y_n \mid n \geq 0\} \subset X$, there exists a unique bounded solution $\{x_n \mid n \geq 0\} \subset X$.

By the Banach theorem on inverse operator, there exists a linear continuous operator $C : l_\infty(X) \to l_\infty(X)$, which transforms an arbitrary bounded sequence $\{y_n \mid n \geq 0\} \subset X$ into the corresponding bounded solution $\{x_n \mid n \geq 0\} \subset X$ of Eq. (1).

By contradiction, assume that there exists $z \in \sigma(A)$, $|z| \leq 1$. Then one of the following two cases is possible:

1. z is an eigenvalue of the operator A:

$$\exists v \in X, \;\; v \neq \overline{0}: \quad Av = zv.$$

Since $|z| \leq 1$, the sequence $\left\{A^n v : n \geq 1\right\}$ is bounded. By Lemma 2, we get

$$\|R(n,-1)v\| = \|R(n,-1)A^{-n}A^n v\| \leq L\,\|A^n v\|, \quad n \geq -1.$$

This means that the sequence $\{R(n-1,-1)v : n \geq 0\}$ is bounded. However, this sequence is a solution of the homogeneous equation (1). Since the operator $R(n-1,-1)$ has the inverse, this solution is nontrivial. We arrive at a contradiction.

2. z is not an eigenvalue of the operator A but there exists an element $v \in X$ such that

$$\forall x \in X: \quad (A - zI)x \neq v.$$

We rewrite Eq. (1) in the form

$$x_{n+1} = Ax_n + (A_n - A)x_n + y_n, \quad n \geq 0. \tag{3}$$

By using Lemma 1, we can find $N \in \mathbf{N}$ and a bounded sequence $\{y_n \mid n \geq 0\} \subset X$ such that

$$(A_n - A)(Cy)_n + y_n = z^n v, \quad n \geq N.$$

The bounded solution $x = Cy \in l_\infty(X)$ of Eq. (3) corresponding to this sequence satisfies the equation

$$x_{n+1} = Ax_n + z^n v, \quad n \geq N. \tag{4}$$

This equation cannot have another bounded solution because the fact that the sequence $\left\{A^n x_N : n \geq N\right\}$ is bounded leads to a contradiction as in Case 1.

However,

$$\left\{w_n = z^{-1} x_{n+1} : n \geq N\right\}$$

is also a bounded solution of Eq. (4). Therefore, $x_n = z^{-1} x_{n+1}$, $n \geq N$. Thus, we get $x_n = z^{n-N} x_N$, $n \geq N$. Substituting this in (4), we obtain

$$z^{1-N} x_N = Az^{-N} x_N = v \Leftrightarrow (A - zI)\left(-z^{-N} x_N\right) = v.$$

We arrive at a contradiction.

Sufficiency. Assume that $\sigma(A) \subset \{z \in \mathbf{C} \mid |z| > 1\}$. This implies that the spectral radius of the operator A^{-1} is smaller than 1. Hence,

$$\exists C > 0 \ \exists \varepsilon \in (0,1) \ \forall n \in \mathbf{N}: \quad \|A^{-n}\| \leq C(1-\varepsilon)^n.$$

By using Lemma 2, we get

$$\|R(m,n)\| \leq L\,\|A^{m-n}\| \leq LC(1-\varepsilon)^{n-m}, \quad n > m \geq -1.$$

Thus, for any bounded solution $\{x_n \mid n \geq 1\}$, by using the representation from Lemma 2, we get

$$x_0 + R(-1,0)y_0 + R(-1,1)y_1 + \ldots + R(-1,n-1)y_{n-1} = R(-1,n-1)x_n \to \overline{0}, \quad n \to \infty.$$

Hence, the element

$$x_0 = -\sum_{k=0}^{\infty} R(-1,k)y_k$$

is determined uniquely. The solution starting from this element is bounded due to the estimate

$$\|x_n\| = \left\|\sum_{k=n}^{\infty} R(n-1,k)y_k\right\| \leq \|y\|_\infty \sum_{k=n}^{\infty} LC(1-\varepsilon)^{k-n+1} \leq \frac{LC}{\varepsilon}\,\|y\|_\infty.$$

Example 1. In the space $X = C([0,1])$ with uniform norm, we consider the operators

$$(A_n x)(t) = e^{t+a}(1 + t^n - t^{n+1/n})x(t), \quad t \in [0,1], \quad n \geq 1,$$

where $a \in \mathbf{R}$ is a constant. These operators have the uniform limit A given by the equality

$$(Ax)(t) = e^{t+a}x(t), \quad t \in [0,1].$$

Indeed, in view of the fact that the function $g(t) = t^n - t^{n+1/n}$ takes the maximum value on the segment [0, 1] at the point

$$t = \left(\frac{n^2}{n^2+1}\right)^n,$$

we get

$$\|A_n - A\| = \max_{t\in[0,1]} \left|t^n - t^{n+1/n}\right| = \left(\frac{n^2}{n^2+1}\right)^{n^2} - \left(\frac{n^2}{n^2+1}\right)^{n^2+1} = r_n.$$

Moreover,

$$r_n = \frac{1}{n^2+1}\left(\frac{n^2}{n^2+1}\right)^{n^2} \sim \frac{e^{-1}}{n^2} \to 0, \quad n \to \infty.$$

For the sequence $\{A_n : n \geq 1\}$, we check the remaining conditions of belonging to the class $As(X)$. All mentioned operators have inverse operators and are commuting. Moreover,

$$\left\|A_n A^{-1}\right\| = \max_{t\in[0,1]} \left|1 + t^n - t^{n+1/n}\right| = 1 + r_n,$$

$$\left\|A A_n^{-1}\right\| = \max_{t\in[0,1]} \left|(1 + t^n - t^{n+1/n})^{-1}\right| = 1.$$

Since the infinite product $\prod_{n=1}^{\infty}(1 + r_n)$ is convergent, conditions (v) and (vi) are satisfied.

The spectrum of the operator A is the segment $\left[e^a, e^{1+a}\right]$ of a real straight line in the complex plane.

By virtue of Theorem 3, Eq. (1) has a unique bounded solution for any known bounded sequence if $a > 0$. In the other cases, the condition of existence and uniqueness is not true.

Equation with Initial Condition

Theorem 4. *Suppose that the sequence $\{A_n : n \geq 1\}$ belongs to the class $As(X)$ and has the limit $A \in L(X)$. The difference equation (1) has a bounded solution $\{x_n \mid n \geq 1\} \subset X$ for any bounded sequence $\{y_n \mid n \geq 0\} \subset X$ and any $x_0 \in X$ if and only if*

$$\sigma(A) \subset \{z \in \mathbf{C} \mid |z| < 1\}. \tag{5}$$

Proof. *Necessity.* Assume that, for any $x_0 \in X$ and any bounded sequence $\{y_n \mid n \geq 0\} \subset X$, the solution $\{x_n \mid n \geq 1\} \subset X$ is bounded.

1. Let $y_n = \overline{0}$, $n \geq 0$, and let $x_0 \in X$ be arbitrary. In this case, the analyzed solution takes the form

$$x_n = R(n-1,-1)x_0, \quad n \geq 1.$$

In view of its boundedness, we get

$$\forall x_0 \in X: \quad \sup_{n\geq 0} \|R(n-1,-1)x_0\| < +\infty.$$

By the Banach–Steinhaus theorem, we obtain

$$\sup_{n\geq 0} \|R(n-1,-1)\| < +\infty. \tag{6}$$

2. Let $y_n = R(n,-1)z_0$, $n \geq 0$, and let $x_0 \in X$ and $z_0 \in X$ be arbitrary. In this case, the solution has the form

$$x_n = R(n-1,-1)(x_0 + nz_0), \quad n \geq 1.$$

Since the solution is bounded and condition (6) is satisfied, we have

$$\forall z_0 \in X: \quad \sup_{n\geq 1} \left\|nR(n,-1)z_0\right\| < +\infty. \tag{7}$$

By the Banach–Steinhaus theorem, we get

$$\sup_{n\geq 1} \|nR(n,-1)\| < +\infty. \tag{8}$$

3. Let

$$y_n = (n+1)R(n,-1)z_0, \quad n \geq 0,$$

and let $x_0 \in X$ and $z_0 \in X$ be arbitrary. This sequence is bounded by virtue of (8). By induction, we easily show that the solution takes the form

$$x_n = R(n-1,-1)\left(x_0 + \frac{(n+1)(n+2)}{2}\, z_0\right), \quad n \geq 0.$$

Since the solution is bounded and relation (6) is true, we have

$$\forall y_0 \in X: \quad \sup_{n\geq 1} \left\|n^2 R(n,-1)\right\| < +\infty.$$

By using the conditions from the definition of the class $As(X)$ and the inequalities

$$\|A^n\| = \left\|R(n-1,-1)A_0^{-1}A \ldots A_{n-1}^{-1}A\right\| \leq \|R(n,-1)\| \prod_{k=0}^{n-1} \left\|A_k^{-1}A\right\|, \quad n \geq 1,$$

we conclude that the sequence $\{n^2 A^n : n \geq 1\}$ is bounded.

This implies that, for any $z \in \mathbf{C}$, $|z| \geq 1$, the series

$$\sum_{n=0}^{\infty} \|z^{-n} A^n\|$$

is convergent and, hence, the inverse operator $(A - zI)^{-1} \in L(X)$ exists.

Therefore, the condition $\sigma(A) \subset \{z \in \mathbf{C} \mid |z| < 1\}$ is satisfied.

Sufficiency. Let condition (5) be satisfied. This implies that the spectral radius of the operator A is smaller than 1. Hence,

$$\exists C > 0 \;\; \exists \varepsilon \in (0,1) \;\; \forall n \geq 0: \quad \|A^n\| \leq C(1-\varepsilon)^n.$$

By using Lemma 2, we get

$$\|R(n,m)\| \leq L\|A^{n-m}\| \leq LC(1-\varepsilon)^{n-m}, \quad n \geq m \geq -1.$$

Therefore, the solution of Eq. (1) is bounded for any $x_0 \in X$:

$$\begin{aligned}
\|x_n\| &\leq \|R(n-1,-1)x_0\| + \left\| \sum_{k=0}^{n-1} R(n-1,k) y_k \right\| \\
&\leq CL(1-\varepsilon)^n \|x_0\| + \|y\|_\infty \sum_{k=0}^{n-1} CL(1-\varepsilon)^{n-1-k} \\
&\leq \frac{CL}{\varepsilon} \|y\|_\infty + CL\|x_0\|, \quad n \geq 1.
\end{aligned}$$

Example 2. In the space $X = C([0,1])$ with uniform norm, we consider the operators

$$(A_n x)(t) = e^{t+a}(1 + t^n - t^{n+1/n}) x(t), \quad t \in [0,1],$$

where $a \in \mathbf{R}$ is a constant. These operators have the uniform limit A given by the equality

$$(Ax)(t) = e^{t+a} x(t), \quad t \in [0,1].$$

As shown in Example 1, the sequence $\{A_n : n \geq 1\}$ belongs to the class $As(X)$ and the spectrum of the operator A is the segment $\left[e^a, e^{1+a}\right]$ of the real straight line in the complex plane.

By virtue of Theorem 4, Eq. (1) has a unique bounded solution for any known bounded sequence if $a < -1$. In the other cases, the condition of existence is not true.

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